Sec 7.3 The eigenvalue method for vector ODEs

Homogeneous eq $\underline{X}' = \underline{A}\,\underline{X}$. How do we find the solutions?

Idea: Build solutions from eigenvalues/vectors.

- If $\underline{v}$ has $\underline{A}\underline{v} = \lambda \underline{v}$, set $\underline{X}(t) = e^{\lambda t}\underline{v}$
- Since $\underline{v}$ "constant", $\underline{X}' = (e^{\lambda t})'\underline{v} = \lambda e^{\lambda t}\underline{v}$
- Also, $\underline{A}\,\underline{X} = \underline{A}(\underbrace{e^{\lambda t}}_{\text{scalar}}\underline{v}) = e^{\lambda t}\underline{A}\underline{v} = e^{\lambda t}(\lambda \underline{v}) = \lambda e^{\lambda t}\underline{v}$
- Thus $\underline{X} = e^{\lambda t}\underline{v}$ is a solution for $\underline{X}' = \underline{A}\,\underline{X}$.

If $\underline{X}$ $n\times 1$, $\underline{A}$ $n\times n$, then we need n lin indep solutions $\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_n$ to make general solution

$$\underline{X} = c_1\underline{X}_1 + \cdots + c_n\underline{X}_n.$$

★ Trick/Fact: If $\lambda_1 \neq \lambda_2$, then their eigenvectors $\underline{v_1}$, $\underline{v_2}$ are automatically lin indep. and the functions

$$\underline{X}_1 = e^{\lambda_1 t}\underline{v_1}, \quad \underline{X}_2 = e^{\lambda_2 t}\underline{v_2}$$

are automatically linearly indep.

Part I: Distinct* $\mathbb{R}$ eigenvalues (*aka, all have mult. 1)

Ex Find the general solution for

$$\underline{X}' = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}\underline{X} \qquad \left(\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

Step 1 Compute eigenvalues:

$$\det(\underline{A}-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ -4 & 6-\lambda & 2 \\ +16 & -15 & -5-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 6-\lambda & 2 \\ -15 & -5-\lambda \end{vmatrix} = (3-\lambda)\big((6-\lambda)(-5-\lambda)+30\big)$$

$$= (3-\lambda)(\lambda^2-\lambda)$$

$$= \lambda(3-\lambda)(\lambda-1) = 0$$

so $\lambda=0$, $\lambda=1$, $\lambda=3$ all with multiplicity 1

Step 2 Compute eigenvectors

$\underline{\lambda=0}$ Solve $(\underline{A}-0I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underline{0}$

$$\left[\begin{array}{ccc|c} \cancel{3}\,1 & 0 & 0 & 0 \\ -4 & 6 & 2 & 0 \\ 16 & -15 & -5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cancel{6}\,3 & \cancel{2}\,1 & 0 \\ 0 & \cancel{-15}\,3 & \cancel{-5}\,1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow a=0 \\ \leftarrow 3b+c=0 \\ b=-\frac{1}{3}c \end{array}$$

solutions are $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{3}c \\ c \end{bmatrix} = c\begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$.

pick $c=3 \implies$ set $\underline{v_1} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ is an eigenvector.

$\underline{\lambda_2 = 1}$ Solve $(\underline{A} - I)\underline{v} = \emptyset$

$$\begin{bmatrix} 2 & 0 & 0 \\ -4 & 5 & 2 \\ 16 & -15 & -6 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \emptyset$$

"trick"

$R_1 \Rightarrow 2a = 0 \Rightarrow a = 0$

$R_2 \Rightarrow -4a + 5b + 2c = 0$

$5b + 2c = 0$

$b = -\frac{2}{5}c$ c free

so $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{5}c \\ c \end{bmatrix} = c\begin{bmatrix} 0 \\ -2/5 \\ 1 \end{bmatrix}$. Pick $c = 5$, $\underline{v_2} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$.

$\underline{\lambda_3 = 3}$ ... $v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Step 3: In total, general solution is

$$\underline{X} = c_1\underline{X}_1 + c_2\underline{X}_2 + c_3\underline{X}_3$$

$$= c_1 e^{\lambda_1 t}\underline{v_1} + c_2 e^{\lambda_2 t}\underline{v_2} + c_3 e^{\lambda_3 t}\underline{v_3}$$

(vector form)

$$= c_1 e^{0t}\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

scalar form of solution

$$\begin{cases} x_1(t) = 0 + 0 + c_3 e^{3t} \\ x_2(t) = -c_1 - 2c_2 e^{t} + 0 \\ x_3(t) = 3c_1 + 5c_2 e^{t} + 2c_3 e^{3t} \end{cases}$$

☆ We know $\underline{X}_1, \underline{X}_2, \underline{X}_3$ are linearly indep solutions because $\underline{v_1}$, $\underline{v_2}$, $\underline{v_3}$ are eigenvectors with different e-values.

Part II Distinct complex-conjugate pairs

Review: When solving something like

$$y'' + 4y' + 13y = 0,$$

we found roots (of $r^2 + 4r + 13 = 0$)

$$r = 2 \pm 3i \quad \text{(a root pair w/ multiplicity 1)},$$

and made solution

$$y(x) = c_1 \underbrace{e^{2x}\cos(3x)}_{y_1} + c_2 \underbrace{e^{2x}\sin(3x)}_{y_2}$$

independent sols :

One way we got these was using $e^{ix} = \cos(x) + i\sin(x)$ on

$$c_1 e^{(2+3i)x} + c_2 e^{(2-3i)x}$$

and collecting cos, sin terms.

Another, equivalent way to get $y_1 = e^{2x}\cos(3x)$, $y_2 = e^{2x}\sin(3x)$

was/is using half of the pair, just $2+3i$, writing

$$e^{(2+3i)x} = e^{2x}\cos(3x) + i\, e^{2x}\sin(3x),$$

and setting

$$y_1 = \mathrm{Re}\left(e^{(2+3i)x}\right) = e^{2x}\cos(3x)$$
$$y_2 = \mathrm{Im}\left(e^{(2+3i)x}\right) = e^{2x}\sin(3x)$$

to be our two "basis solutions". We'll follow this latter way in vector ODEs.

$\underline{Ex}$ Find a general solution of system

$$\begin{cases} x_1' = -x_2 \\ x_2' = x_1 \end{cases} \qquad \begin{pmatrix} = 0x_1 - 1x_2 \\ = 1x_1 + 0x_2 \end{pmatrix}$$

This is the same as solving vector ODE

$$\underline{X}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \underline{X}.$$

$\underline{\text{Step 1}}$ $\det(\underline{A} - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$\lambda = \pm i$ (conjugate root pair has mult. 1)

$\underline{\text{Step 2}}$ Find eigenvectors

$\underline{\lambda_1 = +i}$ Solve $(\underline{A} - iI)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -i & 0 \\ -i & -1 & 0 \end{array}\right]$$

Can row reduce with complex numbers $(R_2 \to R_2 + iR_1)$

$$\left[\begin{array}{cc|c} \boxed{1} & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} &-1 + i(-i) \\ &-1 + 1 = 0 \end{aligned}$$

↑ b free

$\Rightarrow a - ib = 0 \Rightarrow a = ib$

so eigenvectors are $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ib \\ b \end{bmatrix} = b\begin{bmatrix} i \\ 1 \end{bmatrix}$, so pick $b = 1$

and set $\underline{v_1} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

★ We could now compute eigenvector $\underline{v_2}$ for $\lambda_2 = -i$, but there's a shortcut: since $\lambda_2 = -i$ is the complex conjugate of λ_1 (i.e., $\lambda_2 = \overline{\lambda_1} = \overline{i} = -i$), automatically an eigenvector for λ_2 is just $\overline{\underline{v_1}}$, i.e.

$$\underline{v_2} = \overline{\underline{v_1}} = \overline{\begin{bmatrix} i \\ 1 \end{bmatrix}} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

could then do $c_1 e^{\lambda_1 t}\underline{v_1} + c_2 e^{\lambda_2 t}\underline{v_2}$, collect cos/sin and real/imag...

Now, to make independent solutions $\underline{X}_1, \underline{X}_2$, we use "half" of the pair $\lambda = \pm i$:

a solution of ODE $\underline{X}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\underline{X}$ is

$$e^{\lambda_1 t}\underline{v_1} = e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} ie^{it} \\ e^{it} \end{bmatrix}$$

Break this into Real/Imaginary parts:

$$\begin{bmatrix} ie^{it} \\ e^{it} \end{bmatrix} = \begin{bmatrix} i(\cos t + i\sin t) \\ \cos t + i \sin t \end{bmatrix}$$

$$= \begin{bmatrix} -\sin t + i\cos t \\ \cos t + i \sin t \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}}_{\underline{X}_1 = \text{real part}} + i \underbrace{\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}}_{\underline{X}_2 = \text{imag part}}$$

so general solution is $c_1\underline{X}_1 + c_2\underline{X}_2$

$$\underline{X} = \begin{bmatrix} -c_1 \sin t + c_2 \overset{\cos}{\cancel{\sin}} t \\ c_1 \cos t + c_2 \sin t \end{bmatrix} \longleftrightarrow \begin{cases} x_1 = -c_1 \sin t + c_2 \cos t \\ x_2 = c_1 \cos t + c_2 \sin t \end{cases}$$

vector form

__Ex__ Find general solution for system

$$\begin{cases} x_1' = 7x_1 + 7x_2 + 5x_3 \\ x_2' = -4x_1 - 4x_2 - 7x_3 \\ x_3' = 4x_1 + 4x_2 + 7x_3 \end{cases}$$

(find eigenvalues λ by "inspection")

$$\underline{X}' = \begin{bmatrix} 7 & 7 & 5 \\ -4 & -4 & -7 \\ 4 & 4 & 7 \end{bmatrix} \underline{X}$$

$$\det(\underline{A} - \lambda I) = \begin{vmatrix} 7-\lambda & 7 & 5 \\ -4 & -4-\lambda & -7 \\ 4 & 4 & 7-\lambda \end{vmatrix}$$

would be painful to do "in full"

remember determinant tricks!

$R_3 \to R_3 + R_2$ does not change determinant!

$$\begin{vmatrix} 7-\lambda^{+} & 7 & 5 \\ -4^{-} & -4-\lambda & -7 \\ 0^{+} & -\lambda^{-} & -\lambda^{+} \end{vmatrix}$$

row 3 $\longrightarrow$

$$= 0 - (-\lambda)\begin{vmatrix} 7-\lambda & 5 \\ -4 & -7 \end{vmatrix} + (-\lambda)\begin{vmatrix} 7-\lambda & 7 \\ -4 & -4-\lambda \end{vmatrix}$$

$$= (-\lambda)\Big[-\big((-7)(7-\lambda) + 20\big) + \big((7-\lambda)(-4-\lambda) + 28\big)\Big]$$

$$= (-\lambda)\Big[-\big(7\lambda - 49 + 20\big) + \big(\lambda^2 + 4\lambda - 7\lambda - 28 + 28\big)\Big]$$

$$= (-\lambda)\Big[-7\lambda + 29 + \lambda^2 - 3\lambda + 0\Big]$$

$$= (-\lambda)\Big[\underbrace{\lambda^2 - 10\lambda + 29}\Big] = 0$$

$\lambda = 0$ (mult 1) $\qquad \lambda = \dfrac{10 \pm \sqrt{100 - 116}}{2} = 5 \pm 2i$ (mult 1) so then need eigenvectors

$\underline{\lambda = 0}$ Solve $(\underline{A} - 0I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underline{0}$

$$\begin{bmatrix} 7 & 7 & 5 \\ -4 & -4 & -7 \\ 4 & 4 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(R_3 \to R_3 + R_2)\left[\begin{array}{ccc|c} 7 & 7 & 5 & 0 \\ -4 & -4 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$(R_1 \to R_1 + 2R_2)\left[\begin{array}{ccc|c} +1 & +1 & +9 & 0 \\ -4 & -4 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$(R_2 \to R_2 + 4R_1)\left[\begin{array}{ccc|c} 1 & 1 & 9 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow a + b + 9c = 0 \\ \leftarrow \underline{c = 0} \quad \Rightarrow \underline{a = -b} \end{array}$$

so eigenvectors are $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix}$; pick $b = 1$, set $\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\lambda = 5 + 2i}$ Solve $(\underline{A} - (5+2i)\underline{I})\underline{v} = \underline{0}$

$$\left[\begin{array}{ccc|c} 2-2i & 7 & 5 & 0 \\ -4 & -9-2i & -7 & 0 \\ 4 & 4 & 2-2i & 0 \end{array}\right]$$

we can "skip" echelon form sometimes with "tricks/observations"

$$(R_1 \leftrightarrow R_3)\left[\begin{array}{ccc|c} 4 & 4 & 2-2i & 0 \\ -4 & -9-2i & -7 & 0 \\ 2-2i & 7 & 5 & 0 \end{array}\right]$$

$$(R_2 \to R_2 + R_1)\left[\begin{array}{ccc|c} 4 & 4 & 2-2i & 0 \\ 0 & -5-2i & -5-2i & 0 \\ 2-2i & 7 & 5 & 0 \end{array}\right] \leftarrow (-5-2i)b + (-5-2i)c = 0$$

$$\Rightarrow b + c = 0$$
$$\Rightarrow \underline{b = -c}$$

$$R_1 \Rightarrow 4a + 4b + (2-2i)c = 0$$
$$4a - 4c + 2c - 2ic = 0 \to 4a - 2c - 2ic = 0$$

$$\boxed{a = \frac{(1+i)c}{2}}$$

So eigenvectors for $\lambda_2 = 5+2i$ are $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1+i}{2} \cdot c \\ -c \\ c \end{bmatrix}$ (c free),

pick $c = 2 \longrightarrow \underline{v_2} = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}$.

Automatically, for $\lambda_3 = 5-2i = \overline{5+2i} = \overline{\lambda_2}$, we know an eigenvector $\underline{v_3}$ for λ_3 will be $\overline{\underline{v_2}} = \overline{\begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}} = \begin{bmatrix} 1-i \\ -2 \\ 2 \end{bmatrix} = \underline{v_3}$

Now for solutions $\underline{X}_1, \underline{X}_2, \underline{X}_3$:

- for $\lambda_1 = 0$, $\underline{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \underline{X}_1 = e^{\lambda_1 t}\underline{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

- for conjugate pair $\lambda = 5 \pm 2i$, we use "half of the pair", specifically $\underline{\lambda = 5+2i}$, eigenvector $\underline{v} = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}$,

make $e^{\lambda t}\underline{v} = e^{(5+2i)t}\begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix} = e^{5t}\begin{bmatrix} (1+i)e^{2it} \\ -2e^{2it} \\ 2e^{2it} \end{bmatrix}$

(use Euler's formula)

$$= e^{5t}\begin{bmatrix} (1+i)(\cos(2t) + i\sin(2t)) \\ -2\,(\cos(2t) + i\sin(2t)) \\ 2\,(\cos(2t) + i\sin(2t)) \end{bmatrix}$$

$$= e^{5t}\begin{bmatrix} \cos(2t) - \sin(2t) + i(\cos(2t) + \sin(2t)) \\ -2\cos(2t) + i(-2\sin(2t)) \\ 2\cos(2t) + i(2\sin(2t)) \end{bmatrix}$$

split "real" and "imaginary" vectors

$$\underbrace{e^{5t}\begin{bmatrix} \cos(2t) - \sin(2t) \\ -2\cos(2t) \\ 2\cos(2t) \end{bmatrix}}_{\underline{X}_2 = \mathrm{Re}(e^{\lambda t}\underline{v})} + i\,\underbrace{e^{5t}\begin{bmatrix} \cos(2t) + \sin(2t) \\ -2\sin(2t) \\ 2\sin(2t) \end{bmatrix}}_{\underline{X}_3 = \mathrm{Im}(e^{\lambda t}\underline{v})}$$

Thus, general solution is

$$\underline{X} = c_1 \underline{X}_1 + c_2 \underline{X}_2 + c_3 \underline{X}_3$$

$$= \begin{bmatrix} -c_1 \\ c_1 \\ 0 \end{bmatrix} + e^{5t} \begin{bmatrix} (c_2+c_3)\cos(2t) + (-c_2+c_3)\sin(2t) \\ -2c_2\cos(2t) - 2c_3\sin(2t) \\ 2c_2\cos(2t) + 2c_3\sin(2t) \end{bmatrix}$$